The Ergodic Theory of Orbit Equivalence Classification of Group Actions

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Measure preserving actions

- (X, μ) is a standard non-atomic measure space.
- $\mathrm{Aut}\,(X, \mu)$ is the group of **measure preserving transformations**: invertible $T : X \to X$ s.t. $\mu \circ T^{-1} = \mu$.
- A **measure preserving action** $G \curvearrowright (X, \mu)$ is a group-homomorphism $G \to \mathrm{Aut}\,(X, \mu)$.
- A measure preserving action is **ergodic** if there are no non-trivial invariant subsets: if $g.A \subset A$ for all g then $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.
- $G, H \curvearrowright (X, \mu)$ are **orbit equivalent** (o.e.) if there exists $T \in \mathrm{Aut}\,(X, \mu)$ s.t. $T(G.x) = H.T(x)$ for almost every x.
- In other words, the orbit equivalence relations of G and H are isomorphic in a measure preserving way.

Orbit classification of measure preserving actions

Theorem (Dye)

All ergodic prob. preserving actions of $\mathbb{Z}$ are o.e.

Theorem (Ornstein & Weiss; Connes, Feldman & Weiss)

All ergodic prob. preserving actions of amenable groups are o.e.

Theorem (Connes & Weiss; Hjorth)

The above theorem is false for all non-amenable groups.

- *Connes & Weiss for non-Kazhdan's property.*
- *Hjorth for Kazhdan's property.*
- *Each non-amenable group has uncountably many non o.e. prob. preserving actions (Ioana, Epstein).*

On the proof of Dye's Theorem

- Key ideas behind Dye's theorem. For ergodic $T \in \mathrm{Aut}(X, \mu)$:

Fact

If $\mu(A) = \mu(B)$, A can be mapped onto B with iterations of T.

Lemma (The Rohlin's Lemma)

For every $N \in \mathbb{N}$ and $\epsilon > 0$, there is $A \subset X$ such that $A, TA, \ldots, T^N A$ are disjoint and together cover X up to ϵ.

- "Proof" of Dye's theorem: if $T \in \mathrm{Aut}(X, \mu)$ and $T' \in \mathrm{Aut}(X, \mu')$ are ergodic, construct a sequence of Rohlin Towers for T and T', refining each other on each level. The corresponding Boolean-mapping gives rise to a point-mapping.
- Ornstein & Weiss generalized Rohlin's Lemma to amenable groups using *tiling*.

Non-singular actions I

- (X, μ) is a standard non-atomic measure space.
- $\mathrm{Aut}\,(X, [\mu])$ is the group of **non-singular transformations**: invertible $T : X \to X$ s.t. $\mu \circ T^{-1}$ and μ are mutually absolutely continuous.
- A **non-singular action** $G \curvearrowright (X, \mu)$ is a group-homomorphism $G \to \mathrm{Aut}\,(X, [\mu])$.
- The notion of **ergodicity** is defined verbatim.
- $G, H \curvearrowright (X, \mu)$ are **orbit equivalent** (o.e.) if there exists $T \in \mathrm{Aut}\,(X, [\mu])$ s.t. $T\,(G.x) = H.T\,(x)$ for almost every x.

Observation

Orbit equivalence and ergodicity depend only on the measure class.

Non-singular actions II

Theorem (Ornstein & Weiss; Connes, Feldman & Weiss)

Every ergodic non-singular action of a countable amenable group is o.e. to a non-singular action of $\mathbb{Z}$.

- For free actions the proof uses Ornstein & Weiss's generalization of Rohlin's Lemma to non-singular actions.
- The non-free case is due to Connes, Feldman & Weiss and is more involved.
- There is a new proof by Andrew Marks that should be interesting to study.

Non-singular actions III (examples)

- Let $X = 2^\omega$ with a probability product measure $\rho = \bigotimes_{n<\omega} \rho_n$ with $\rho_n(0) \in (0,1)$.
- The equivalence relation E_0 on 2^ω is defined by xE_0y if and only if $\#\{n : x(n) \neq y(n)\} < \infty$.
- An action $G \curvearrowright 2^\omega$ is said to be **homoclinic** if its orbit equivalence relation is a sub-relation of E_0.

Example (finite permutations)

Let Π be the group of permutations of ω that change finitely many elements. It has an obvious Følner sequence so it is amenable. The natural action $\Pi \curvearrowright (2^\omega, \rho)$ is non-singular and in many cases it is ergodic (Hewitt-Savage 0-1 law, Aldous-Pitman 0-1 law).

Non-singular actions IV (examples)

Example

$G = \bigoplus_{n<\omega} \mathbb{Z}/2\mathbb{Z}$ with "coordinate-wise addition" acts naturally $G \curvearrowright 2^\omega$ by "flipping" each coordinate.

- Let $\rho = \bigotimes_{n<\omega} (p, 1-p)$. If $p = 1/2$ it is measure preserving, and if $p \neq 1/2$ it is merely non-singular.
- By the Kolmogorov's 0-1 law this action is ergodic w.r.t. ρ.
- Clearly, the orbit equivalence relation of $\bigoplus_{n<\omega} \mathbb{Z}/2\mathbb{Z}$ is E_0.
- $\bigoplus_{n<\omega} \mathbb{Z}/2\mathbb{Z}$ is amenable (either because it is Abelian, or using the obvious Følner sequence), so from the Connes-Feldman-Weiss Theorem it is orbit equivalent to a non-singular action of $\mathbb{Z}$.

Non-singular actions V (examples)

Example (dyadic odometer)

2^ω has a structure of an Abelian group:

- Identify $x=(x(1),\dots,x(n),0,0,\dots)$ with the integer $N(x)=\sum_k x(k)2^k$. If $x, y \in 2^\omega$ are two such elements, $x \oplus y$ is the unique element $z \in 2^\omega$ with $N(x)+N(y)=N(z)$.
- $(1,1,0,0,\dots) \oplus (1,1,1,0,0,\dots) = (0,1,0,1,0,0,\dots)$.
- This rule extends to all of 2^ω and there are inverses.
- For $\mathbf{1}=(1,0,0,\dots)$ we have $\ominus\mathbf{1}=(1,1,1,\dots)$.
- Let $\mathcal{O} : 2^\omega \to 2^\omega$, $\mathcal{O}x = (1,0,0,\dots) \oplus x$. It can be shown that its orbit equivalence relation is E_0.
- Thus, $\mathcal{O}$ and $\bigoplus_{n<\omega} \mathbb{Z}/2\mathbb{Z}$ are o.e. as non-singular actions.

Non-singular actions VI (examples)

Example (shift)

Let G be a countable group and $X = 2^G$. The *shift* $G \curvearrowright 2^G$ is

$$g : x(h) \mapsto x(gh).$$

- The orbit equivalence relation of the shift, denoted by $E(G, 2)$, is far from being homoclinic.
- In the next week talk I will discuss the very interesting relations between the shift and the homoclinic actions.

Krieger's classification I

- Two cases for a non-singular ergodic action $G \curvearrowright (X, \mu)$:
  1. The action is *essentially* measure preserving: there is a measure ν on X s.t. (i) ν is equivalent to μ (ii) ν is G-invariant.
     - Thus, $G \curvearrowright (X, \mu)$ is isomorphic, in the non-singular category, to the measure preserving action $G \curvearrowright (X, \nu)$.
  2. The action is *genuinely* non-singular: there is no ν as before.
- The first case corresponds to the classical theory of measure preserving actions and Dye's theorem provides a full answer (caution: depending on whether ν is finite or infinite)
- The second case is different and requires new ideas.

Krieger's classification II

- An ergodic non-singular action is **Type** II if it is essentially measure preserving, or **Type** III if it is genuinely non-singular.
- This terminology originates in the classification of factors in operator algebras: an ergodic non-singular action has an associated factor von Neumann-algebra, and o.e. actions have isomorphic von Neumann-algebras.

Theorem (Krieger's Classification Theorem)

An ergodic non-singular Type III *action of amenable groups can be further classified into Types* III_λ, $0 \leq \lambda \leq 1$, *such that*

- *Type* III_λ, $\lambda \in (0, 1]$, *is a complete invariant of o.e.*
- *Type* III_0 *contains many o.e. classes of its own.*

Radon-Nikodym cocycle

Definitions (essential values, ratio set)

Let $G \curvearrowright (X, \mu)$ be a non-singular action.

1. The **R-N cocycle** $\psi : G \times X \to \mathbb{R}$ is $\psi_g(x) = \log \frac{d\mu \circ g}{d\mu}(x)$. It is a cocycle in the sense that
$$\psi_{gh}(x) = \psi_g(h.x) + \psi_h(x).$$
2. $r \in \mathbb{R}$ is an **essential value** for G if for all $A \subset X$, $\mu(A) > 0$, and $\epsilon > 0$, there can be found $g \in G$ with
$$\mu\left(A \cap g^{-1}(A) \cap \{|\psi_g - r| < \epsilon\}\right) > 0.$$
3. The **ratio set** $\mathrm{e}(G, \mu)$ is the set of all essential values.

The Ratio Set

Lemma

The ratio set $e(G,\mu)$ *of a non-singular ergodic action* $G \curvearrowright (X,\mu)$ *is a non-empty closed subgroup of* $\mathbb{R}$*. Hence it is either of:*

- $\mathrm{e}(G,\mu) = \{0\}$ *for* ***Type*** III_0*;*
- $\mathrm{e}(G,\mu) = \mathbb{Z}\log\lambda$ *for* ***Type*** III_λ *with* $\lambda \in (0,1)$*; and*
- $\mathrm{e}(G,\mu) = \mathbb{R}$ *for* ***Type*** III_1*.*

- Two o.e. actions have the same Type (technical but elementary). The converse is hard.
- Krieger showed that there is a more delicate abstract invariant, called **the associated flow**, which will not be discussed here.

Dyadic odometer

Example

Let $\mathcal{O} : 2^\omega \to 2^\omega$ be the odometer, $\rho = \bigotimes_{n<\omega}(p, 1-p)$, $p \neq 1/2$.

- Let the cylinder $C_n = [1, \ldots, 1, 0]_1^n \subset 2^\omega$.
- For $x \in C_n$, $\mathcal{O}x = (0,\ldots,0,1,x(n+1),\ldots)$ because $1 + \sum_{k=1}^{n-1} 2^k = 2^n$.
- For $x \in C_n$,

$$\frac{d\mu \circ \mathcal{O}}{d\mu}(x) = \frac{\mu\left([0,\ldots,0,1]_1^n\right)}{\mu\left([1,\ldots,1,0]_1^n\right)} = \frac{p^{n-1}(1-p)}{(1-p)^{n-1}p} = \left(\frac{p}{1-p}\right)^{n-2}$$

  so the R-N cocycle is $\psi_{\mathcal{O}}(x) = (n-2)\log\frac{p}{1-p}$ on C_n.
- $e(\mathcal{O}, \rho) = \mathbb{Z}\log\frac{p}{1-p}$ so the odometer is Type $\mathrm{III}_{\frac{p}{1-p}}$.

Finite permutations

Example

Let $\Pi \curvearrowright (2^{\omega}, \rho)$ with $\rho = \bigotimes_{n<\omega} \rho_n$, $\rho_n(0) \in (0,1)$.

- For transposition $\pi : i \leftrightarrow j$, $\frac{d\rho\circ\pi}{d\rho}(x) = \frac{\rho_i(x_j)\rho_j(x_i)}{\rho_i(x_i)\rho_j(x_j)}$.
- If $\lim_{k\to\infty}\rho_{n_k}(0)=p$, $\lim_{j\to-\infty}\rho_{n_j}(0)=q$ then $\log\frac{p}{1-p}\frac{1-q}{q} \in e(\Pi,\rho)$.
- Take a cylinder C supported on $[1,N]$ and $\epsilon > 0$.
- Fix n_k, n_j large and $B = C \cap \{x : x(n_k) = 0, x(n_j) = 1\}$.
- $\pi : n_k \leftrightarrow n_j$ satisfies $\pi(B) \subset C$ and for $x \in B$,

$$\frac{d\rho\circ\pi}{d\rho}(x) = \frac{\rho_{n_k}(1)\,\rho_{n_j}(0)}{\rho_{n_k}(0)\,\rho_{n_j}(1)} \approx \frac{p}{1-p}\frac{1-q}{q}.$$

Thank you